Control Laws for Optimal Spacecraft Navigation

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A set of formulas for the calculation of optimal space flight trajectories is presented. The flight takes place in the central gravitational field of a celestial body, with low or high thrust from the rocket propulsion engine. The formulas analyze the optimal control of the thrust direction using polar coordinates and take the rocket engine's optimal switch conditions for the free-flight periods of the spacecraft into consideration. Trajectories in optimal time and fuel consumption are studied. By these formulas, the trajectory calculation becomes a solution of a two-point boundary value problem of a differential equation system. Finally, a control law for a nearly optimal change in the angle of inclination of the space flight trajectory is demonstrated.

Nomenclature

c = exhaust velocity
g = gravitational acceleration, $= \gamma/r^2$
H = Hamiltonian
M = rocket mass
m = rate of fuel consumption
r = radius
s = thrust acceleration, $= cm/M$
u, v = spacecraft velocity
α = thrust angle
β = thrust angle
γ = gravitational constant
ϑ = inclination angle
λ = Lagrange multiplier
φ = path angle

Introduction

THIS paper introduces analytical and computational simplifications for the optimal control of rocket-powered spacecraft. It is assumed that the travel takes place in the central gravitational field and the motion of the spacecraft is exclusively influenced by the thrust of its rocket engine and gravitational forces. The Lagrange method is applied to the trajectory optimization, which leads to a boundary value problem for a system of differential equations. Lawden[1] has given a detailed description of the application of this method to spacecraft trajectory optimization. The problem is improved by replacing the Lagrange multipliers with thrust angles and their time derivatives. This solution is outlined in the following sections.

Effect of Gravitational Field

Since the spacecraft moves in the space that surrounds a celestial body, its position in a plane can be described by the polar coordinates r (radius) and φ (path angle), with a pole at the center of attraction. See Fig. 1.

Introducing the velocity components u (in the radial direction) and v (perpendicular to r), the motion can be formulated by the following system of first-order differential equations:

$$\dot{u} = \frac{v^2}{r} - \frac{\gamma}{r^2} + \frac{c \cdot m}{M} \sin\beta \tag{1}$$

$$\dot{v} = -\frac{u \cdot v}{r} + \frac{c \cdot m}{M} \cos\beta \tag{2}$$

$$\dot{r} = u \tag{3}$$

$$\dot{\varphi} = v/r \tag{4}$$

$$\dot{M} = -m \tag{5}$$

where γ is the gravitational constant and the effective exhaust velocity c is assumed to be constant. No use will be made of Eq. (4). A dot above a formula sign indicates the total differentiation with respect to time $\mathrm{d}/\mathrm{d}t$.

The well-known Lagrange method can now be applied; following this method, the Hamiltonian H is constructed as

$$H = \lambda_u \left(\frac{v^2}{r} - \frac{\gamma}{r^2} + \frac{c \cdot m}{M} \sin\beta \right) + \lambda_v \left(-\frac{u \cdot v}{r} + \frac{c \cdot m}{M} \cos\beta \right) + \lambda_r u - \lambda_M m \tag{6}$$

In this function, the courses of the Lagrange multipliers are given by the Lagrange equations, which here take the form of

$$\dot{\lambda}_u = -\frac{\partial H}{\partial u} = \frac{v}{r}\lambda_v - \lambda_r \tag{7}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = -2\frac{v}{r}\lambda_u + \frac{u}{r}\lambda_v \tag{8}$$

$$\dot{\lambda}_r = -\frac{\partial H}{\partial r} = \left(\frac{v^2}{r^2} - 2\frac{\gamma}{r^3} \right)\lambda_u - \frac{u \cdot v}{r^2}\lambda_v \tag{9}$$

$$\dot{\lambda}_M = -\frac{\partial H}{\partial M} = \frac{c \cdot m}{M^2}(\lambda_u \sin\beta + \lambda_v \cos\beta) \tag{10}$$

The optimal thrust angle control is determined by

$$\frac{\partial H}{\partial \beta} = \frac{c \cdot m}{M}(\lambda_u \cos\beta - \lambda_v \sin\beta) = 0 \tag{11}$$

or

$$\tan\beta = \lambda_u/\lambda_v$$

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Besides the direction of thrust, the spaceship can be controlled by its thrust level. Generally, the thrust of a rocket engine (given by the product of fuel consumption m and effective exhaust velocity c) is constant. Such is often the case for rocket engines using chemical propellants. A decrease in the rate of propellant expenditure m will throttle the motor, but this raises technical problems and trajectories flown by throttled engines are usually nonoptimal. So-called "singular thrust arcs" demand a control of the rate m in the following form:

$$\frac{\partial H}{\partial m} = \frac{c}{M}(\lambda_u \sin\beta + \lambda_v \cos\beta) - \lambda_M = 0 \qquad (12)$$

The control law [Eq. (12)] can be used indirectly for a calculation of the optimal function $m(t)$. Lawden[1] has given a complete analytical evaluation of these nonoptimal singular thrust arcs. The optimal trajectories consist of flight periods using maximum thrust—"thrust-flight periods"—and periods with the engine off—"free-flight periods."

According to the Pontryagin's principle, the Hamiltonian H and the Lagrange multipliers λ should be continuous at instants of discontinuous control.[2] If the multipliers do not vanish simultaneously, it follows from this condition that the thrust angle β and its first time derivative $\dot\beta$ are continuous along the optimal trajectory. Nevertheless, the mass flow rate m may be discontinuous, for example, at instants when the engine is switched. A continuous course of the Hamiltonian and the Lagrange multipliers demands the following conditions at the switching instants:

$$H = \left(\frac{v^2}{r} - \frac{\gamma}{r^2}\right)\lambda_u - \frac{u \cdot v}{r}\lambda_v + u\lambda_r \qquad (13)$$

$$0 = \frac{c}{M}(\lambda_u \sin\beta + \lambda_v \cos\beta) - \lambda_M \qquad (14)$$

Use of the Weierstrass-Erdmann corner conditions will lead to the same switch conditions.[2]

The value H in Eq. (13) depends on the flight time restrictions. H is well known to be constant if the time t does not occur explicitly in H. In case the flight time is not predetermined, H vanishes, according to the Lagrange theory, at the final point of the trajectory and therefore on the whole flight path. Of course, flight-time-unrestricted trajectories ($H = 0$) demand less fuel consumption than trajectories with predetermined time of arrival ($H \neq 0$).

The problem now is to calculate the initial values of the Lagrange equations (7–10) in such a way that the spaceship reaches a desired terminal point after having flown its transfer trajectory. A flight path controlled by these equations maximizes the final value of the rocket mass M.

The following statements are well known in spacecraft flight mechanics:

1) The flight path depends very sensitively on the initial values of the Lagrange equations. Small errors in these values lead to extremely different trajectories.

2) The switch conditions [Eqs. (13) and (14)] are nearly everywhere valid on the optimal trajectory.

3) The behavior of the differential equation system is highly nonlinear.

Hence, it is difficult to solve these equations; thus, it may be better to replace them by an adequate set of control laws. An analytical elimination of the Lagrange multipliers leads to formulas that can be clearly represented by a small number of variables.

Gravitational and Thrust Accelerations

By introducing the gravitational acceleration $g = \gamma / r^2$ and the thrust acceleration $s = cm/M$, the equations of motion (1-5) can be replaced by two second-order differential equations[3]

$$\text{Radial: } \ddot r - r\dot\varphi^2 = s \sin\beta - g \qquad (15)$$

$$\text{Horizontal: } r\ddot\varphi + 2\dot r\dot\varphi = s \cos\beta \qquad (16)$$

The thrust angle β should be controlled optimally. Equations (7–9) and (11) provide a set of four equations to calculate β. They can be replaced by one second-order differential equation.

$$r(\ddot\beta - \ddot\varphi) + 2\dot r(\dot\beta - \dot\varphi) - 3g \sin\beta \cos\beta + 2r(\dot\beta^2 - 3\dot\beta\dot\varphi + 2\dot\varphi^2)\tan\beta = 0 \qquad (17)$$

employing differentiation and the equations of motion.[4] This control law is a necessary condition for a time optimal or fuel-consumption optimal thrust angle control; it gives the final value of spacecraft mass M a relative maximum. Especially, Eq. (17) is valid for every arbitrary course of thrust acceleration s, even in cases of singular thrust arcs and even if s is not continuous. During the thrust-flight periods, thrust acceleration s increases in time, because the spacecraft looses fuel mass. But the working rocket engine must be switched off at the instant when the following switch-off condition has been reached:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot r}{r \tan\beta} - \dot\varphi\right) = 0 \qquad (18)$$

This condition is equivalent to Eq. (13), using Eqs. (7) and (8) and putting $H = 0$. It is valid as an equation during the free-flight period that follows and, during that time period, it defines the thrust angle β as an integral of Eq. (17). Of course, there is no necessity to control the thrust angle when the motor is not operative; in any case, the spacecraft moves along a conic orbit (often an elliptic orbit). But, the optimal switch-on instant can be calculated by using Eq. (18) as a control law, because the beginning and final instants of the free-flight period are connected by Eq. (14). After eliminating the Lagrange multipliers, it takes the following form:

$$\left(\frac{\dot r}{\sin\beta}\right)\text{beginning} = \left(\frac{\dot r}{\sin\beta}\right)\text{final} \qquad (19)$$

In this notation, "beginning" and "final" refer to the beginning and final times for the free-flight periods.

A determination of this switch-on condition demands the computation of thrust angle β during the free-flight time; β and its first time derivative $\dot\beta$ are continuous everywhere. The flight time of a transfer trajectory controlled by these switch conditions is fuel-consumption optimal with regard to small variations. Of course, it is possible to reduce the transfer time at the expense of a higher fuel consumption. For example, such problems may occur when a rendezvous maneuver with a

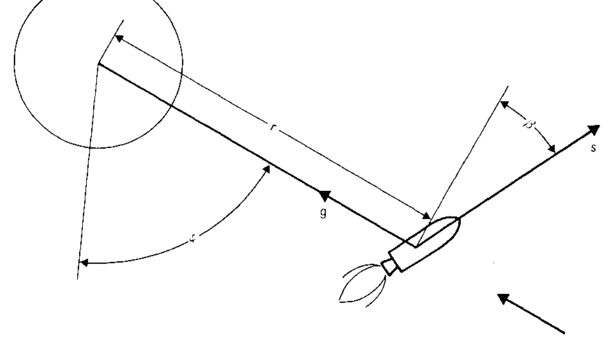

Fig. 1 Spacecraft in the gravitational field.

target object has to be flown. Trajectories with reduced flight times also require a control of the thrust angle β by Eq. (17); but $H \neq 0$ and the rocket motor burns longer (dependent on the entire transfer time) than predetermined by Eq. (18). After the free-flight period, the motor is switched on again and a new switch-on condition becomes valid,

$$\left[\frac{r \sin^2\beta}{\cos\beta}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{r\tan\beta}-\dot{\varphi}\right)\right]\text{beginning} = \text{final} \qquad (20)$$

Thus, by the switch conditions—in contrast to the thrust angle control law, time optimization is not the same as fuel-consumption optimization.

If equations of motion (15) and (16) determine the spacecraft trajectory as a function of its control and if the thrust angle control law [Eq. (17)], together with the switch conditions [Eqs. (18) and (19)], determine the course of the control, then the terminal point reached by the spacecraft depends upon only the initial values of the differential equations. If the initial orbit of the spacecraft is given, the transfer trajectory and, thus, the final destination orbit of the spacecraft depend upon only the initial values β and $\dot{\beta}$ of differential equation (17). Unfortunately, there is no possibility to analyze the initial conditions analytically as a function of the destination orbit, for there is no analytical solution of the equations of the motion during the thrust-flight periods. The final trajectory calculation must be done numerically on a computer, which takes initial trial values and integrates the trajectory. From deviations in the final values of the differential equations, the initial values can be adjusted. The integration process must be repeated until the final values are satisfied.[5]

For numerical computer solutions, Eq. (17) is ill-conditioned. Introducing z,

$$z = \frac{r^2\dot{\beta} - rv}{\cos^2\beta} \qquad (21)$$

second-order differential equation (17) can be transformed into two first-order differential equations as

$$\dot{\tan\beta} = \frac{z}{r^2} + \frac{v}{r}\left(\tan^2\beta + 1\right) \qquad (22)$$

$$\dot{z} = \frac{\tan\beta}{r}\left(4vz + 3\gamma\right) \qquad (23)$$

and the control variables β and $\dot{\beta}$ are replaced by the variables $\tan\beta$ and z. Equations (22) and (23) must be adjoined to the equations of motion (1-5). Together, they provide a system of first-order differential equations that is better conditioned for application on a computer than the original set of equations. As an example for such a numerical computer solution, Fig. 2 shows the flight path of a Space Shuttle upper stage. Starting from a circular low Earth orbit, the upper stage carries a satellite to a (coplanar) geostationary target orbit. The control function $\tan\beta$ of the trajectory is given in Fig. 3.

The integration of the trajectory includes integration of state equations (1-5) for the given initial values (r_0, u_0, v_0, φ_0, M_0) and the control equations.

To calculate the control, either the transformed equations (22) and (23) or the Lagrange equations (7-10) can be used. The initial values of the control equations can be derived from β_0 and $\dot{\beta}_0$ as follows:

$$(\tan\beta)_0 = \tan\beta_0$$

$$z_0 = \frac{r_0^2\dot{\beta}_0 - r_0 v_0}{\cos^2\beta_0} \qquad (24)$$

or

$$\lambda v_0 = 1, \; \lambda u_0 = \tan\beta_0 \qquad (25)$$

$$\lambda r_0 = \frac{v_0}{r_0}\left(1 + 2\tan^2\beta_0\right) - \frac{u_0}{r_0}\tan\beta_0 - \frac{\dot{\beta}_0}{\cos^2\beta_0}$$

Then, the integration of the trajectory for a predetermined flight time t yields the same final values for the state variables, whether the transformed equations (22) and (23) or the original equations (7-9) are used. The three original equations need about three times as many numerical calculation operations than the two transformed equations. Also, generally, the behavior of the variables is the same. The required numerical calculation time for the integration of the control diminishes by about 50% using the transformed system (and, therefore, the trajectory calculation time by about 20%).

To show in a particular instance that the transformations are proper, numerical comparison calculations were performed. Some of the results are given in Fig. 4. The behavior of the variables is shown to be smooth and the calculation time is reduced by the use of the transformed system.

Change of Inclination Angle

If the destination orbit is not coplanar with the initial orbit, the spacecraft has to change the inclination angle while per-

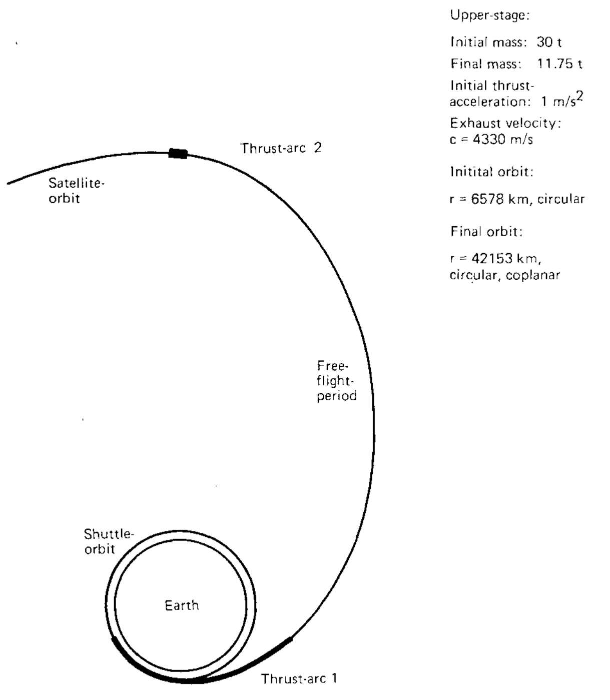

Fig. 2 Flight path of a Space Shuttle upper stage.

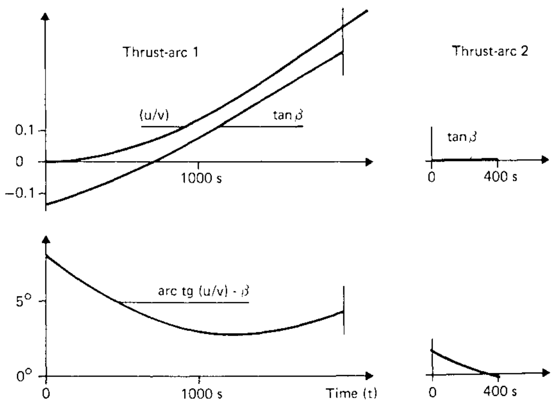

Fig. 3 Trajectory control function of the flight path.

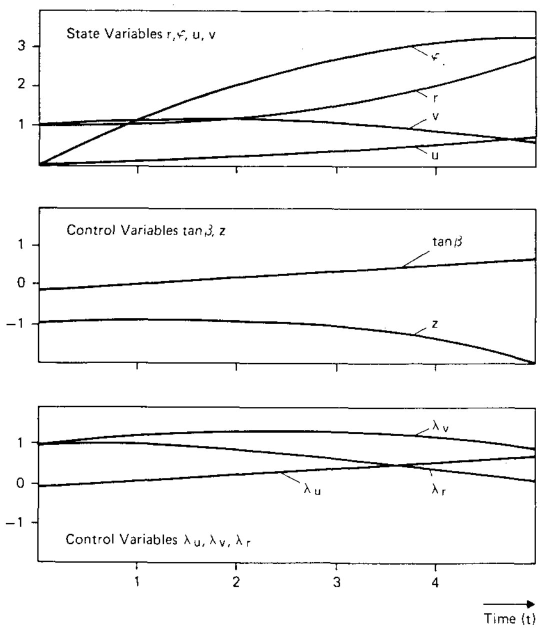

Fig. 4 Numerical comparison calculations.

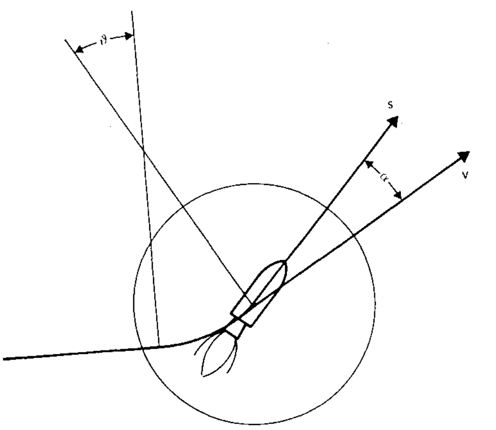

Fig. 5 Change in the inclination angle.

forming its orbital transfer. Imagine the thrust acceleration vector s sloped out of the trajectory plane (picture plane, Fig. 1) in such a way that orthogonal projection of this vector lies upon the original vector. By sloping the vector out of the trajectory plane, the thrust acceleration component in the trajectory plane is diminished by $\cos\alpha$ if the angle between the thrust vector and the trajectory plane is α. The course of the thrust acceleration does not directly affect the control law defining the course of thrust angle β; hence, Eq. (17) is still valid if the control program of thrust angle α is a given function of time.

Component $s \sin\alpha$ of the thrust acceleration vector sloped out of the trajectory plane, creating a rotation of this trajectory plane and thus an inclination change. By this, the course of thrust angle α should be optimized. The Hamilton-Lagrange theory, combined with the elimination of the Lagrange multipliers, applied to plausible simplified equations of motion shows that a very simple control law for thrust angle α is nearly optimal.

First, the motion of the spacecraft in this maneuver should be formulated by equations. Imagine the spacecraft trajectory seen from a certain point in space, as if standing on the arrow in Fig. 1. The gravitational center (the celestial body) lies behind the spacecraft. From this point of view, its instantaneous path might look like Fig. 5.

The change of inclination angle ϑ should be completed when the spacecraft is far beyond the gravitational center, for there its velocity v is small and the path angle φ does not change rapidly far from the gravitational center. If the thrust acceleration is not too low, the path angle can be assumed to be constant. The rotation velocity $\dot{\vartheta}$ of the trajectory plane is then simply the component $s \sin\alpha$ of the thrust acceleration vector divided by the horizontal component of spacecraft velocity v. The value of v can be calculated by equating the acceleration $\dot{v}$ with the thrust acceleration $s \cos\alpha$ that lies in the trajectory plane, because the radial components of thrust and velocity can be neglected in the apse of the transfer ellipse. Thus, inclination angle ϑ and spacecraft velocity v may be found by integrating the following differential equations:

$$\dot{\vartheta} = s \sin\alpha / v \tag{26}$$

$$\dot{v} = s \cos\alpha \tag{27}$$

The course of the thrust angle α is determined by the Hamiltonian

$$H = \lambda_\vartheta \frac{s \sin\alpha}{v} + \lambda_v \cdot s \cos\alpha \tag{28}$$

together with the Lagrange equations

$$\dot{\lambda}_\vartheta = -\frac{\partial H}{\partial \vartheta} = 0 \tag{29}$$

$$\dot{\lambda}_v = -\frac{\partial H}{\partial v} = \lambda_\vartheta \frac{s \sin\alpha}{v^2} \tag{30}$$

and the optimal control law

$$\frac{\partial H}{\partial \alpha} = \lambda_\vartheta \cos\alpha \frac{s}{v} - \lambda_v s \sin\alpha = 0 \tag{31}$$

The elimination of the Lagrange multipliers λ_ϑ and λ_v brings us to the following control law:

$$\dot{\alpha} + \frac{s}{v} \sin\alpha = \dot{\alpha} + \dot{\vartheta} = 0 \tag{32}$$

As ϑ increases, α must be diminished. The plausible result is that, in relation to an inertial system, the thrust angle is constant. At the beginning of the inclination change during the thrust-flight period, the initial value of differential equation (32) has to be chosen recognizing the fact that at the end of that period the desired inclination change is realized. A flight path controlled by this equation may provide a good initial estimate of an accurate numerical computer solution of a fully optimized three-dimensional spacecraft trajectory.

Conclusion

The Hamilton-Lagrange theory applied to the equations of motion of a spacecraft that moves with the high or low thrust of the engine through the central gravitational field of a celestial body leads to a set of nonlinear control equations. Improvement of the problem is possible by an analytical elimination of the Lagrange multipliers in the control equations: not only is the number of variables reduced, but so is the number of control laws. Therefore, numerical results can be obtained more easily and more accurately than with schemes using the more direct application of the Lagrange method.

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